MATHEMATIX CLUB

NISER Mathematics Club

SUMS 123

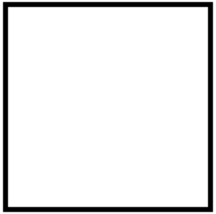
CHOP ME IF YOU CAN!

Sandipan Samanta & Aaditya Vicram Saraf

February 7, 2025

A square?

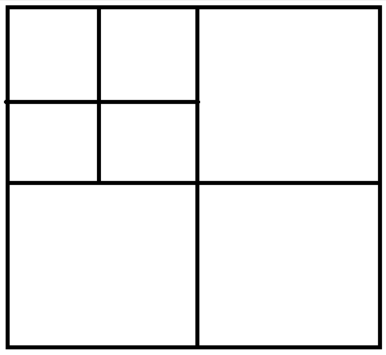
Our task today is to chop a given square into smaller squares in the way of a partition. We seek to find that what are the possible number of smaller squares which can be hence obtained.



Dividing a square into k-many squares

Problem is to determine that how many squares can we obtain. For example, we can trim our square into 4 smaller squares (easy to check) or 7 smaller squares as shown.

Problem Statement: Given a square, find all the possible values of k such that the square can be partitioned (or chopped) into $k-$many smaller squares.



Trivially we will never talk about chopping into 1 square because it is basically doing nothing. Further, all results have been done considering the unit square. We leave the case for $k = 2$ as an easy exercise and start with the case where $k = 3$.

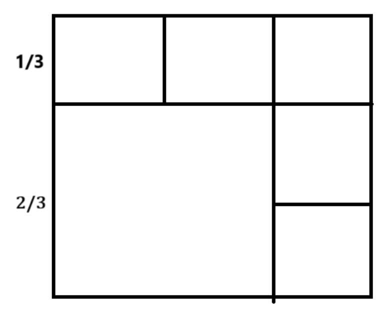
Before we get there, we propose the claim.

Claim

For any even $k \geq 4$, we can chop our square into k smaller squares.

Proof.

Let $k = 2m$ for some $m \geq 2$. Then divide one side of the square into m parts each of length $\frac{1}{m}$. With that we obtain $2m$ squares each of side length $\frac{1}{m}$ but there's an overlap, hence we only get $2m - 1$ distinct squares. The remaining portion itself is a square of sidelength $\frac{m-1}{m}$. Hence total we get $2m$ many smaller squares.



□

Claim

For any odd $k \geq 7$, we can chop our square into k smaller squares.

Proof.

Let $k = 2m + 1$ for some $m \geq 3$.

Now, note that we can chop a square into 4 squares. So, if we chop any of the smaller square into 4 squares, we are basically increasing the value of k by 3. Hence, if we can divide a square into $2m - 2 = 2(m - 1)$ squares then we can divide any of one of the smaller square into 4 parts and that will give a partition into $2m - 2 + 3 = 2m + 1$ squares.

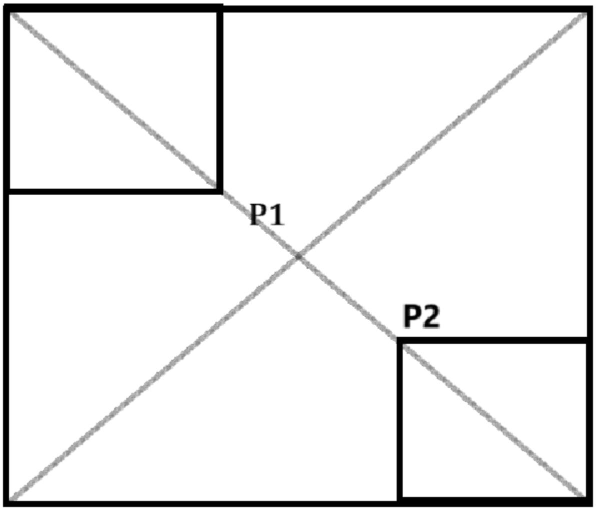
But, with the previous claim, we know how to divide a square into $2(m - 1)$ squares where $m \geq 3$. Thus, it is also possible to divide it into $2m + 1$ squares.

□

Above proofs show that for every $k \geq 6$ we can chop our squares into k squares. Only numbers left now, are 2, 3 and 5. Now we are mature enough to try the case for $k = 3, 5$.

Case I : $k = 3$

It is crucial to observe that if we chop our squares into smaller squares, it is necessary that each of the 4 corners are part of some square. Now notice that if it happens, then for diagonally opposite points the squares which they belong have one point along a main diagonal of our bigger square. (This happens because the main diagonal is also the angle bisector of the first corner, and since we must get a square, the diagonal of smaller square attached with the first corner must be overlapping with the bigger square's diagonal). Name these points as $P1, P2$ (as shown in diagram below). Consider when $P1 = P2$. In this case we obtain 4 squares, not 3, and when $P1 \neq P2$ then we cannot make any further cuts (since we already are left with 3 parts). But now we have an angle which is greater than 180^o. This is clearly not a square, hence our claim holds.


P1
P2

Case II : $k = 5$

We treat this case similar as the case for $k = 3$ but we need to be a little more careful here. We observe that all 4 corners are parts of some squares.
For better visualization we have named some points of interest (diagram on the next slide). Consider the simple case where all 4 points are distinct. In this case we get 5 parts (hence we cannot cut further) and we also obtain reflex angles (i.e angles greater than 180^o). Thus the remaining part is not a square and we cannot cut further, hence we do not get 5 smaller squares in this way.
Then we consider the case when at least one of the pairs $(P1, P2)$ or $(P3, P4)$ are same, i.e the case when $P1 = P2$ or $P3 = P4$. WLOG assume that $P1 = P2$. Then we obtain 4 smaller squares, and there are 2 squares which contain the points $P3$ and $P4$ respectively.

Apart from the squares obtained from points $P1$ and $P2$, we need 3 more squares which should be obtained from the remaining 2 squares. This means we need to cut one square into 2 squares which is not possible. $\square$

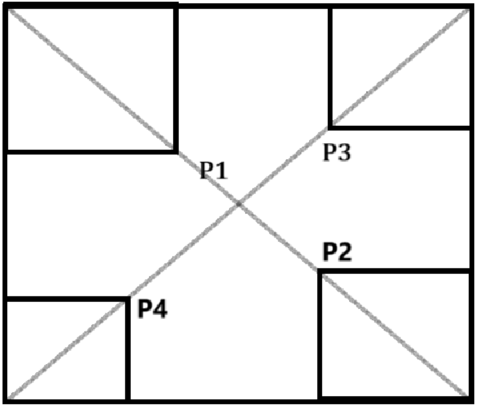


Figure 1: $P1 \neq P2$ & $P3 \neq P4$

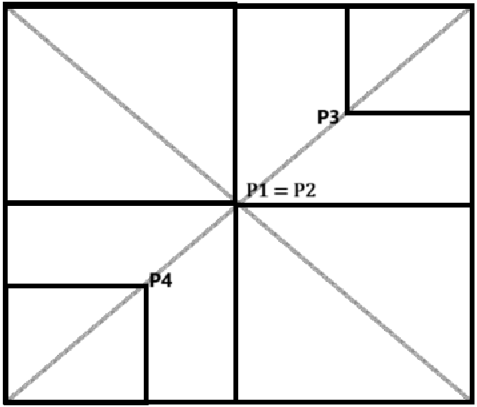
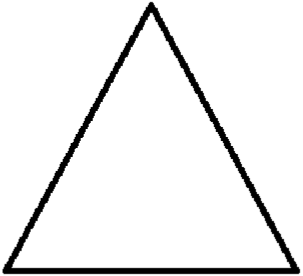


Figure 2: $P1 = P2$

A triangle?

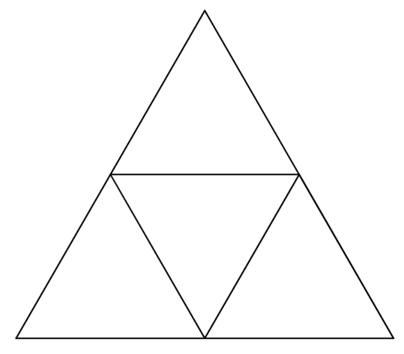
Let us convert the same problem into that of an equilateral triangle. We seek to chop it into smaller equilateral triangles, and see how many smaller triangles are obtainable.



Dividing an equilateral triangle into k-many equilateral triangle

Convention : We only talk about equilateral triangles hence the word 'equilateral' will be omitted.

Problem Statement: Given a triangle, find all the possible values of k such that the triangle can be partitioned (or chopped) into $k-$many smaller triangles.

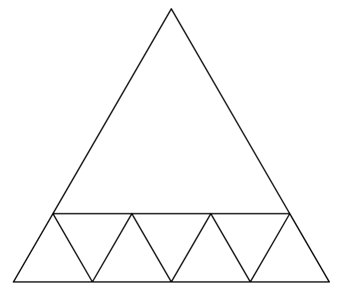


Claim

For any even $k \geq 4$, we can chop our triangle into k smaller triangles.

Proof.

Let $k = 2m$. Now divide one side of the triangle into m equal parts. (Refer to the case when $m = 4$ in the next page). These will give us smaller m equilateral triangles and the other $m - 1$ vertically inverted triangles. Then only 1 leftover part remains which itself is an equilateral triangle. Thus we get $m + m - 1 + 1 = 2m$ triangles.



□

Claim

For any odd $k \geq 7$, we can chop our triangle into k smaller triangles.

Proof.

Let $k = 2m + 1$ for some $m \geq 3$.

Now, note that we can chop a triangle into 4 triangles. So, if we chop any of the smaller triangle into 4 traingles, we are basically increasing the value of k by 3. Hence, if we can divide a triangle into $2m - 2 = 2(m - 1)$ triangle then we can divide any of one of the smaller triangle into 4 parts and that will give a partition into $2m - 2 + 3 = 2m + 1$ triangle.

But, with the previous claim, we know how to divide a triangle into $2(m - 1)$ triangles where $m \geq 3$. Thus, it is also possible to divide it into $2m + 1$ traingles.

□

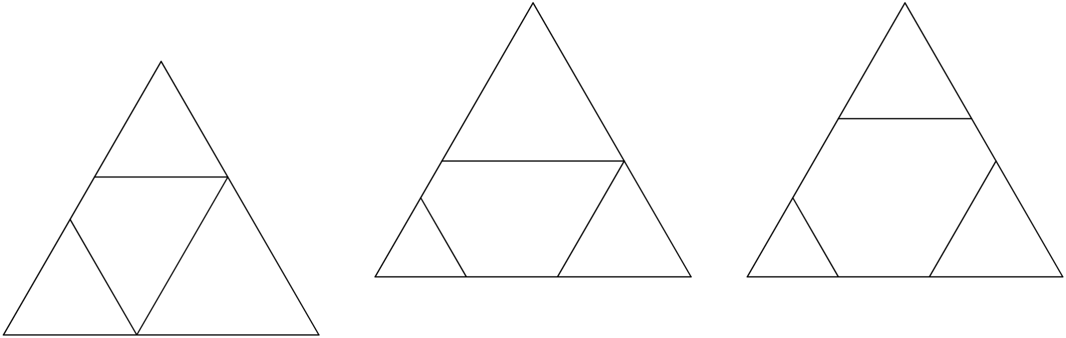
Above proofs show that for every $k \geq 6$ we can chop our triangles into k triangles. Only numbers left now, are 2, 3 and 5. Let us see if we get a similar result like the previous one? (We again leave the case for $k = 2$ as an exercise).

Case I : $k = 3$

Like in the case of a square, every vertex must be a part of a smaller triangle. If we have exactly 3 triangles, consider removing one triangle, then we will be left with a trapezium which is not a parallelogram. But if this remaining polygon gives us 2 further triangles, it means on joining two triangles, we must obtain this trapezium. Since we have two equilateral triangles, joining them will yield only a parallelogram (or if it is not joined side to side then we get some reflex angles in our polygon so obtained). Thus we cannot divide our triangle into 3 smaller triangles. □

Case II : $k = 5$

We treat this case similar as the case for $k = 3$ but we need to be a little more careful here.
If we cut out 3 triangles, each associated with some vertex, then whatever shape we are left with, we need to extract 2 triangles from it. There are few possible cases, as shown in Fig. 18 apart from which the case which remains is where the remaining part is a triangle. In that case we cannot cut it further to obtain two more triangles.
Like the argument for the case of $k = 3$, the case when we get a trapezium can again be safely rejected. To reject the case where the remaining part is a pentagon or a hexagon, we can use the argument that since by joining two triangles we can only obtain a parallelogram (otherwise we will be left with a non-convex polygon).



Observation

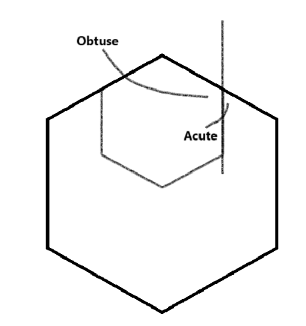
On realizing that for both, a regular 3-gon and a regular 4-gon, we can only chop it into k-smaller regular n-gons (where $n = 3, 4$) is when $k \neq 2, 3, 5$. This tempted us to make a generatisation that can we say this for all regular $n-$gons. An extremely simple and elegant answer, resolved this.

Claim

A regular n-gon cannot be chopped into any number of regular $n-$gons for $n \geq 5$.

Proof.

Consider a regular n-gon. Then each vertex is part of the smaller n-gons into which it is chopped. Consider a regular n-gon respective to some vertex, then it implies we would have to make a cut on an edge connected to that vertex. Angles of a regular n-gon for $n \geq 5$ are obtuse angles, hence if a cut is made through an edge, then it creates an acute angle as well on that edge. But then on further cutting this acute angle must also be a part of a regular n-gon, which is not possible.



□

Thank You

References

- Idea : `https://www.geeksforgeeks.org/puzzle-dividing-a-square-into-n-smaller-squares/`, along with great contribution from one of our seniors (Mr. Ayanava Mandal)