Why do we like Primes?

A brief overview of twin primes, Bertrand's lemma, and related results

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Table of Contents

Prime Densities

Observation

Define M_n to be product of all primes $\leq n$, then it was discovered that there is always a prime between n and $M_n + 1$.

Proof is simple. Since for all primes $p \leq n$, $p | M_n$
$p \nmid M_n + 1$
So $M_n + 1$ must have a prime factor greater than n.
$\implies \exists$ prime p such that $n < p \leq M_n + 1$

Bertrand's postulate

Statement

For every $n \in \mathbb{N},\ n > 1,\ \exists$ a prime p such that :

$$n < p < 2n$$

.

Another formulation, where p_n is the n^{th} prime is :

$$\text{for } n \geq 1,\ p_{n+1} < 2p_n$$

Introduction to Twin Primes

We are probably well familiar with primes. Twin primes are nothing but a pair of primes which have a difference 2, i.e $p_2 = p_1 + 2$.

$$\text{Examples : } (3,5),(5,7),(11,13),(17,19),\ldots$$

Digital root of product of twin primes

Digital root

Take the sum of digits of a number. If it is greater than 10, repeat the process. Example :
Digital root of 69420 $\rightarrow 6+9+4+2+0=21 \rightarrow 2+1=3$

Digital root of product of twin primes

Consider (5,7)
$5 \times 7 = 35,\ 3 + 5 = 8$

Similarly, (17,19)
$17 \times 19 = 323, 3 + 2 + 3 = 8.$

Again, $(29, 31)$
$29 \times 31 = 899, 8 + 9 + 9 = 26, 2 + 6 = 8$

Is it a coincidence?

Let's go a little large
$(191, 193)$ gives $191 \times 193 = 36863$
$3 + 6 + 8 + 6 + 3 = 26, 2 + 6 = 8.$

Digital root of product of twin primes

Theorem :

Let p $\geq$ 5 and p+2 be twin primes. Digital root of p(p+2) is always 8.

Proof :

For any prime $p \geq 5$,
$2, 3 \nmid p \implies p \equiv 1, -1 \pmod 6$
For twin primes p & $p+2$,
if $p \equiv 1 \pmod 6$, then $p + 2 \equiv 3 \pmod 6 \implies 3 \mid p + 2$
A contradiction.
Hence $p \equiv -1 \pmod 6$ and $p + 2 \equiv 1 \pmod 6$.
So let $p = 6k - 1, p + 2 = 6k + 1$
$p(p + 2) = 36k^2 - 1 \equiv 8 \pmod 9$.

We will prove that residue mod 9 is the digital root for any $n \in \mathbb{N}$.

Twin Prime Conjecture

The Conjecture

There are infinitely many twin primes. They are frequent for smaller integers, but get rarer and rarer as the number increases.

Examples

The biggest known twin primes are $2996863034895 \times 2^{1290000} \pm 1$

Euler Totient Function and Divisor Function

Definition 1

Totient function : $\phi(n)$ is defined as the number of positive integers $\leq n$ such that they are co-prime to n.
Example : $\phi(4) = 2$, $\phi(15) = 8$, $\phi(100) = 40$

Definition 2

Divisor function : $\sigma(n)$ is defined as the sum of all positive divisors of n.
Example : $\sigma(4) = 7$, $\sigma(15) = 24$, $\sigma(100) = 217$

Zumkeller Theorem

Statement :

For twin primes p_2 and p_1, with $p_2 > p_1$,

$$\phi(p_2) = \sigma(p_1)$$

The proof is left as an exercise.

Mersenne and Fermat Primes

Mersenne Primes :

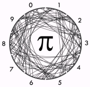
Primes of the form $2^n - 1$.
An interesting and easy to prove result, is that if $2^n - 1$ is a prime, then n is also a prime

Fermat Primes :

Primes of the form $F_n = 2^{2^n} + 1$ are known as Fermat Prime. Fermat conjectured that for all $n \in \mathbb{N}$, F_n is prime, but his claims were proved wrong, when it was discovered that F_5 is not prime. It is interesting that F_1, F_2, F_3, F_4 are the only known Fermat primes. After that till F_{32} which has been discovered, all have turned out to be composite. It is still unknown if there is any other Fermat Primes.
Fun fact : $F_{n+1} = F_0 F_1 \ldots F_{n-1} + 2$

Other related statements

- **Goldbach Conjecture :**
  Every even natural number greater than 2 can be written as the sum of two prime numbers.
- **Legendre's Conjecture :**
  For any $n \in \mathbb{N}$, there is a prime between n^2 and $(n+1)^2$.
- **Oppermann's Conjecture :**
  For any $n \in \mathbb{N}$, there is a prime between $n(n-1)$ and $(n+1)^2$ and at least another prime between n^2 and $n(n+1)$.
- **Prime Number Theorem (PNT) :**
  It tells us about the approximate distribution between primes and gives the prime counting function $\pi(n) \sim \frac{N}{log(N)}$
- **Dubner's Conjecture :**
  It states that every even number greater than 4208 is a sum of twin primes.



THANK YOU