Oscillatory Integrals of First Kind

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Introduction

Oscillatory integrals have played a central role in harmonic analysis since its inception. The Fourier transform itself exemplifies such integrals, and historically, related techniques and ideas can be traced back to the works of Fourier, Airy, Stokes, and Riemann. These early developments often involved tools like Bessel functions and the method of stationary phase, which are now cornerstones in the study of asymptotic behavior.

This report is concerned with *oscillatory integrals of the first kind*, where one studies integrals of the form

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx, \tag{1}$$

as $\lambda \to \infty$, with ϕa real-valued smooth phase function and ψ a smooth (often compactly supported) amplitude. A central objective is to understand the asymptotic behavior of $I(\lambda)$, particularly as influenced by the critical points of ϕ, i.e., the zeros of $\phi'(x) = 0$.

In recent decades, the study of oscillatory integrals has broadened significantly, encompassing a variety of analytic and geometric contexts. Of particular importance is the interaction between oscillatory phenomena and curvature. It is now well-understood that the decay properties of the Fourier transform of a measure supported on a surface are closely tied to the geometric features of the surface—most notably, its curvature. This relationship has deep implications for restriction theory, dispersive estimates, and partial differential equations.

The role of curvature is further exemplified in the analysis of Fourier transforms of surface-supported measures. When the surface has non-vanishing Gaussian curvature, one obtains sharper decay estimates than in the flat case. These considerations lie at the heart of many problems in harmonic analysis and motivate deeper study into curvature-based phenomena.

This direction continues to be a rich area of research, with significant contributions highlighting the subtle interplay between geometry and analysis. Foundational works, such as Elias M. Stein's Problems in Harmonic Analysis Related to Curvature [3], delve into these topics in depth, exploring how curvature influences Fourier analysis on manifolds and how it governs the behavior of oscillatory integrals. The present chapter introduces several fundamental results in this area, setting the stage for more advanced developments.

1 Oscillatory Integrals in One Variable

The one-dimensional theory is well developed, and much of it can be summarized in three guiding principles:

1. **Localization:** The integral's behavior is dominated by the critical points of the phase function.
2. **Scaling:** The decay rate of the integral depends on the order of vanishing of derivatives of ϕ.
3. **Asymptotics:** Near a single critical point, the integral admits a full asymptotic expansion in inverse powers of λ.

1.1 Localization

We begin with the simplest result: if ϕ' does not vanish on the support of ψ, then $I(\lambda)$ decays rapidly.

Proposition 1. *Let $\phi, \psi \in C^\infty(a,b)$, with ψ compactly supported in (a,b), and suppose $\phi'(x) \neq 0$ on $\mathrm{supp}(\psi)$. Then for every integer $N > 0$,*

$$|I(\lambda)| \leq C_N \lambda^{-N}, \quad \text{as } \lambda \to \infty. \tag{2}$$

Proof. Let $D = \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}$ be a differential operator. Note that applying D to $e^{i\lambda\phi(x)}$ gives back the same function:

$$D^N e^{i\lambda\phi(x)} = e^{i\lambda\phi(x)}.$$

Integrating by parts repeatedly, we obtain

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx = \int_a^b D^N(e^{i\lambda\phi(x)})\psi(x)\,dx = \int_a^b e^{i\lambda\phi(x)}({}^tD)^N\psi(x)\,dx,$$

where ${}^tD$ is the transpose of D. Since each application of ${}^tD$ introduces a factor of λ^{-1} and derivatives of ψ and ϕ, and all such functions are smooth and compactly supported, we get

$$|I(\lambda)| \leq C_N\lambda^{-N},$$

for all N, as desired. □

1.2 Scaling

Suppose we are given a phase function ϕ for which a higher derivative is bounded below. We aim to understand how this affects the decay rate of the integral.

Proposition 2 (Van der Corput's Lemma)**.** *Let $\phi \in C^\infty(a,b)$ be real-valued, and suppose there exists an integer $k \geq 2$ such that $|\phi^{(k)}(x)| \geq 1$ on (a,b). Then there exists a constant C_k such that*

$$\left| \int_a^b e^{i\lambda\phi(x)}\, dx \right| \leq c_k \lambda^{-1/k}. \tag{3}$$

When $k = 1$, the same estimate holds provided ϕ' is monotonic.

Proof. The proof is by induction on k.

Base case $k = 1$: If ϕ' is monotonic and $|\phi'(x)| \geq 1$, define

$$D = \frac{1}{i\lambda\phi'(x)} \frac{d}{dx}.$$

Integrating by parts once using D gives boundary terms of size $O(\lambda^{-1})$, and the interior integral reduces to

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq \frac{2}{\lambda} + \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) \right| dx = \frac{2}{\lambda} + \frac{1}{\lambda} \left| \int_a^b \frac{d}{dx} \left(\frac{1}{\phi'(x)} \right) dx \right| \leq \frac{3}{\lambda}.$$

Inductive step: Let us suppose that the case k is known, and assume (replacing ϕ by $-\phi$ if necessary) that

$$\phi^{(k+1)}(x) \geq 1 \quad \text{for all } x \in [a,b]$$

Let $x = c$ be the (unique) point in $[a,b]$ where $\left|\phi^{(k)}(x)\right|$ assumes its minimum value. If $\phi^{(k)}(c) = 0$ then, outside the interval $(c-\delta, c+\delta)$, we have that $\left|\phi^{(k)}(x)\right| \geq \delta$ (and, of course, $\phi'(x)$ is monotonic in the case $k = 1$). Write

$$\int_a^b = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b$$

By our inductive hypothesis,

$$\left|\int_a^{c-\delta} e^{i\lambda\phi}dx\right| \leq c_k(\lambda\delta)^{-1/k} \quad \text{and} \quad \left|\int_{c+\delta}^{b} e^{i\lambda\phi}dx\right| \leq c_k(\lambda\delta)^{-1/k}$$

Since $\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi}dx\right| \leq 2\delta$, we have

$$\left|\int_a^b e^{i\lambda\phi}dx\right| \leq \frac{2c_k}{(\lambda\delta)^{1/k}} + 2\delta$$

If $\phi^{(k)}(c) \neq 0$, and so c is one of the endpoints of $[a,b]$, a similar argument shows that $c_k(\lambda\delta)^{-1/k} + \delta$ is an upper bound for the integral. In either situation, the case $k+1$ follows by taking

$$\delta = \lambda^{-1/(k+1)},$$

which proves (3) with $c_{k+1} = 2c_k + 2$; since $c_1 = 3$, we have $c_k = 5 \cdot 2^{k-1} - 2$.

$\square$

Corollary 3. *Under the assumptions of Proposition 2,*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \leq c_k\lambda^{-1/k}\left(|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right) \tag{4}$$

Proof. This is proved by writing (1) as $\int_a^b F'(x)\psi(x)dx$, with

$$F(x) = \int_a^x e^{i\lambda\phi(t)}dt$$

integrating by parts, and using the estimate

$$|F(x)| \leq c_k\lambda^{-1/k}, \qquad \text{for } x \in [a,b],$$

obtained previously. $\square$

1.3 Asymptotics

When ϕ' vanishes at exactly one point in the support of ψ, and $\phi^{(k)}(x_0) \neq 0$ for some $k \geq 2$, the integral $I(\lambda)$ admits an asymptotic expansion in descending powers of λ.

Proposition 4 (Method of Stationary Phase)**.** *Let $k \geq 2$, and assume*

$$\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(k-1)}(x_0) = 0,$$

but $\phi^{(k)}(x_0) \neq 0$. Suppose ψ is supported in a sufficiently small neighborhood of x_0. Then

$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\, dx \sim \lambda^{-1/k} \sum_{j=0}^{\infty} a_j \lambda^{-j/k} \tag{5}$$

in the sense that for any nonnegative integers N and r,

$$\left(\frac{d}{d\lambda}\right)^r \left[I(\lambda) - \lambda^{-1/k} \sum_{j=0}^{N} a_j \lambda^{-j/k} \right] = O\left(\lambda^{-r-(N+1)/k}\right) \quad \textit{as } \lambda \to \infty. \tag{6}$$

Proof. We begin by proving the result in the simplest case, namely $k = 2$. The argument consists of three main steps.

Step 1. We first observe the asymptotic formula

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} x^\ell e^{-x^2}\, dx \sim \lambda^{-(\ell+1)/2} \sum_{j=0}^{\infty} c_j^{(\ell)} \lambda^{-j} \tag{7}$$

for each nonnegative integer ℓ. If ℓ is odd, the integral vanishes due to symmetry. In fact, the integral equals

$$\int_{-\infty}^{\infty} e^{-(1-i\lambda)x^2} x^\ell\, dx.$$

Setting $z = (1 - i\lambda)^{1/2} \cdot x$, and using the rapid decay of e^{-z^2}, we may deform the contour $(1 - i\lambda)^{1/2} \cdot \mathbb{R}$ back to $\mathbb{R}$. Thus, we find

$$(1 - i\lambda)^{-1/2-\ell/2} \int_{-\infty}^{\infty} e^{-x^2} x^\ell\, dx.$$

Choosing the principal branch of the power $(1 - i\lambda)^{-(\ell+1)/2}$ in the complex plane with a cut along the negative real axis, we obtain

$$(1 - i\lambda)^{-(\ell+1)/2} = \lambda^{-(\ell+1)/2} \cdot (\lambda^{-1} - i)^{-(\ell+1)/2}$$

for $\lambda > 0$. Then a power series expansion of $(w - i)^{-(\ell+1)/2}$ around $w = 0$ (valid for $|w| < 1$) gives the expansion (7), with $w = \lambda^{-1} \to 0$.

Step 2. Next, suppose $\eta \in C_0^\infty$ and ℓ is a nonnegative integer. Then we have the estimate

$$\left| \int_{-\infty}^{\infty} e^{i\lambda x^2} x^\ell \eta(x)\, dx \right| \leq A\lambda^{-1/2-\ell/2}. \tag{8}$$

To prove this, let $\alpha \in C^\infty$ satisfy $\alpha(x) = 1$ for $|x| \le 1$ and $\alpha(x) = 0$ for $|x| \ge 2$, and split the integral as

$$\int e^{i\lambda x^2} x^\ell \eta(x)\,dx = \int e^{i\lambda x^2} x^\ell \eta(x)\alpha(x/\varepsilon)\,dx + \int e^{i\lambda x^2} x^\ell \eta(x)[1-\alpha(x/\varepsilon)]\,dx,$$

where $\varepsilon > 0$ is a small parameter to be chosen.

The first integral is clearly bounded by $C\varepsilon^{\ell+1}$. The second integral is estimated using integration by parts:

$$\int e^{i\lambda x^2} \left({}^tD\right)^N \left\{x^\ell \eta(x)[1-\alpha(x/\varepsilon)]\right\} dx,$$

where ${}^tDf = -\frac{1}{i\lambda}\frac{d}{dx}\left(\frac{f}{2x}\right)$. One can compute that the resulting integral is bounded by

$$\frac{C_N}{\lambda^N}\int_{|x|\ge\varepsilon} |x|^{\ell-2N}\,dx = C_N'\lambda^{-N}\varepsilon^{\ell-2N+1}$$

provided $\ell - 2N < -1$. So altogether, the full integral is bounded by

$$C_N\left[\varepsilon^{\ell+1} + \lambda^{-N}\varepsilon^{\ell-2N+1}\right].$$

Choosing $\varepsilon = \lambda^{-1/2}$ and $N > (\ell+1)/2$ gives the desired estimate.

Similarly, a straightforward integration by parts shows that

$$\int e^{i\lambda x^2} g(x)\,dx = O(\lambda^{-N}), \quad \text{for all } N \ge 0, \tag{9}$$

whenever $g \in \mathcal{S}$ vanishes near the origin.

Step 3. We now prove the proposition for the special case $\phi(x) = x^2$. Consider

$$\int e^{i\lambda x^2}\psi(x)\,dx = \int e^{i\lambda x^2} e^{-x^2}\left[e^{x^2}\psi(x)\right]\tilde{\psi}(x)\,dx,$$

where $\tilde{\psi} \in C_0^\infty$ is identically 1 on the support of ψ. Expand

$$e^{x^2}\psi(x) = \sum_{j=0}^N b_j x^j + x^{N+1}R_N(x) = P(x) + x^{N+1}R_N(x).$$

Substituting into the integral yields three terms:

$$\sum_{j=0}^N b_j \int_{-\infty}^{\infty} e^{i\lambda x^2} e^{-x^2} x^j\,dx \tag{a}$$

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} x^{N+1} R_N(x) e^{-x^2}\tilde{\psi}(x)\,dx \tag{b}$$

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} P(x) e^{-x^2}[\tilde{\psi}(x) - 1]\,dx \tag{c}$$

Apply (7) to (a), (8) to (b), and (9) to (c). The sum of these contributions yields the asymptotic expansion for $I(\lambda)$ in the case $\phi(x) = x^2$.

Now for the general quadratic case. Suppose ϕ satisfies the hypotheses with $k = 2$, so we may write

$$\phi(x) = c(x - x_0)^2 + O(|x - x_0|^3),$$

where $c \neq 0$. This can be expressed as

$$\phi(x) = c(x - x_0)^2[1 + \varepsilon(x)],$$

where $\varepsilon(x) = O(|x - x_0|)$ is smooth and satisfies $|\varepsilon(x)| < 1$ near x_0. Since $\phi'(x) \neq 0$ away from x_0, we can change variables by defining

$$y = (x - x_0)[1 + \varepsilon(x)]^{1/2}.$$

This defines a diffeomorphism from a neighborhood of x_0 to one near $y = 0$, with $\phi(x) = cy^2$. Thus

$$\int e^{i\lambda\phi(x)}\psi(x)\, dx = \int e^{i\lambda cy^2}\tilde{\psi}(y)\, dy,$$

where $\tilde{\psi} \in C_0^\infty$ if ψ is supported near x_0. The expansion (5) for $k = 2$ then follows.

Finally, the argument for higher k follows similar reasoning, using the model integral

$$\int_0^\infty e^{i\lambda x^k} e^{-x^k} x^\ell\, dx = c_{k,\ell}(1 - i\lambda)^{-(\ell+1)/k},$$

and proceeding analogously.

□

Remark 5. *Each constant a_j that appears in the asymptotic expansion* (5) *depends on only finitely many derivatives of ϕ and ψ at x_0. Note that our proof shows that, say in the case $k = 2$, we have*

$$a_0 = \left(\frac{2\pi}{-i\phi''(x_0)}\right)^{1/2} \psi(x_0)$$

Moreover, if k is even, all odd-numbered a_j vanish.

1.4 Example: Bessel Functions

The Bessel function $J_m(r)$ of integral order m is defined by

$$J_m(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin\theta} e^{-im\theta}\, d\theta. \tag{10}$$

This is of the form (1) with $\lambda = r$, and phase function $\phi(x) = \sin x$. Observe that $\phi'(x) = \cos x$ vanishes only at $\pi/2$ and $3\pi/2$ in the interval $[0, 2\pi]$, and there $\phi'' = \pm 1$.

Now choose a partition of unity:

$$1 = \psi_1 + \psi_2 + \psi_3,$$

where ψ_1 is supported near $\pi/2$ and equals 1 near $\pi/2$, and ψ_2 is supported near $3\pi/2$ and equals 1 near $3\pi/2$. The remaining function ψ_3 is supported away from these points.

Substituting this decomposition into equation (10), we write $J_m(r)$ as a sum of three integrals. For the first two terms, where the phase has non-degenerate critical points of order $k = 2$, we apply Corollary 3. For the third term, where the derivative does not vanish, we apply the $k = 1$ case. This yields the estimate

$$J_m(r) = O\left(r^{-1/2}\right), \quad \text{as } r \to \infty.$$

If we instead apply the sharper estimate from Proposition 4 to the first two terms, and use Proposition 1 for the third, we can derive the full asymptotic expansion. The leading term is given by

$$J_m(r) = \left(\frac{2}{\pi}\right)^{1/2} r^{-1/2} \cos\left(r - \frac{\pi m}{2} - \frac{\pi}{4}\right) + O\left(r^{-3/2}\right).$$

More generally, the complete asymptotic expansion of $J_m(r)$ is

$$J_m(r) \sim r^{-1/2} e^{ir} \sum_{j=0}^{\infty} a_j r^{-j} + r^{-1/2} e^{-ir} \sum_{j=0}^{\infty} b_j r^{-j}, \tag{11}$$

for suitable coefficients a_j, b_j.

The Bessel function is also defined for non-integral real values of m. When $m > -1/2$, it is given by

$$J_m(r) = \frac{(r/2)^m}{\Gamma(m + 1/2)\pi^{1/2}} \int_{-1}^{1} e^{irt}(1 - t^2)^{m-1/2} dt. \tag{12}$$

To verify that this agrees with the previous definition (10) when m is a non-negative integer, observe that the identity holds for $m = 0$. Moreover, both definitions satisfy the recurrence relation:

$$\frac{d}{dr}\left[r^{-m} J_m(r)\right] = -r^{-m} J_{m+1}(r).$$

When m is a half-integer, which is common in applications, the integral in (16) can be evaluated by parts to yield elementary functions. In this case, the asymptotic expansion (11) becomes exact, with all terms for $j > m$ vanishing.

2 Oscillatory Integrals in Several Variables

We now turn to integrals of the form

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x)\, dx, \tag{13}$$

where ϕ is a real-valued smooth function defined on an open subset of $\mathbb{R}^n$, and ψ is a smooth function with compact support. As in the one-variable case, we are interested in the behavior of $I(\lambda)$ as $\lambda \to \infty$.

The general principle is that the main contributions to $I(\lambda)$ arise from neighborhoods of critical points of ϕ, i.e., those points where the gradient $\nabla\phi(x) = 0$.

Proposition 6. *Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be compactly supported and $\phi \in C^\infty$ be real-valued, such that $\nabla\phi(x) \neq 0$ on the support of ψ. Then for every integer $N > 0$,*

$$\left| \int e^{i\lambda\phi(x)} \psi(x)\, dx \right| = O(\lambda^{-N}) \quad \text{as } \lambda \to \infty. \tag{14}$$

Proof. For each point x_0 in the support of ψ, there exists a unit vector ξ and a sufficiently small ball $B(x_0)$, centered at x_0, such that

$$\xi \cdot (\nabla\phi)(x) \geq c > 0$$

for all $x \in B(x_0)$, where c is a positive constant.

We decompose the integral $I(\lambda)$ as a finite sum:

$$I(\lambda) = \sum_k \int e^{i\lambda\phi(x)} \psi_k(x)\, dx,$$

where each ψ_k is a smooth function supported entirely within one of the balls $B(x_0)$. It therefore suffices to establish the desired estimate for each individual integral in this sum.

Next, we select a coordinate system $(x_1, x_2, \ldots, x_n)$ so that the first coordinate axis is aligned with the direction of the vector ξ. In these coordinates, we can write

$$\int e^{i\lambda\phi(x)} \psi_k(x)\, dx = \int \left(\int e^{i\lambda\phi(x_1,\ldots,x_n)} \psi_k(x_1, \ldots, x_n)\, dx_1 \right) dx_2 \cdots dx_n.$$

By invoking Proposition 1, we know that the inner integral in x_1 exhibits rapid decay in λ. Therefore, the entire integral is rapidly decreasing, and the desired conclusion follows.

□

2.1 Scaling

Proposition 7. *Suppose ψ is a smooth function supported in the unit ball, and let ϕ be a real-valued function such that, for some multi-index α with $|\alpha| > 0$, we have*

$$|\partial_x^\alpha \phi| \geq 1$$

throughout the support of ψ. Then

$$\left| \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x)\, dx \right| \leq c_k(\phi) \cdot \lambda^{-1/k} \cdot (\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}) \tag{15}$$

where $k = |\alpha|$, and the constant $c_k(\phi)$ is independent of λ and ψ, depending only on bounds for the C^{k+1} norm of ϕ.

Proof. Let us consider the vector space of real homogeneous polynomials of degree k in $\mathbb{R}^n$, which has dimension $d(k, n)$. A natural basis for this space is given by the monomials $\{x^\alpha : |\alpha| = k\}$. It turns out — as will be shown in subsubsection 2.1.1 — that the polynomials

$$(\xi^j \cdot x)^k, \quad j = 1, \ldots, d(k, n)$$

with suitable unit vectors $\xi^1, \ldots, \xi^{d(k,n)}$, also form a basis for this space.

Assuming this fact, suppose there exists a point x_0 in the support of ψ where

$$|\partial_x^\alpha \phi(x_0)| \geq 1$$

for some multi-index α with $|\alpha| = k$. Then, there exists a unit vector $\xi = \xi(x_0)$ such that

$$\left|(\xi \cdot \nabla)^k \phi(x_0)\right| \geq a_k > 0.$$

Moreover, since we assume the C^{k+1} norm of ϕ is bounded, it follows that

$$\left|(\xi \cdot \nabla)^k \phi(x)\right| \geq \frac{a_k}{2}, \quad \text{for } x \in B(x_0),$$

where $B(x_0)$ is a ball centered at x_0 with radius depending only on $\|\phi\|_{C^{k+1}}$.

We now construct a covering of $\mathbb{R}^n$ by such balls of fixed radius, along with a subordinate partition of unity:

$$1 = \sum \eta_j(x), \quad 0 \leq \eta_j \leq 1, \quad \sum |\nabla \eta_j| \leq b_k,$$

where each η_j is supported in one of the balls described above. Define $\psi_j = \psi \cdot \eta_j$, and observe that

$$\int e^{i\lambda\phi} \psi\, dx = \sum \int e^{i\lambda\phi} \psi_j\, dx.$$

To estimate each integral $\int e^{i\lambda\phi}\psi_j\,dx$, choose coordinates so that the first axis aligns with ξ. Then the integral becomes

$$\int e^{i\lambda\phi}\psi_j\,dx = \int \left(\int e^{i\lambda\phi(x_1,\ldots,x_n)}\psi_j(x_1,\ldots,x_n)\,dx_1 \right) dx_2\cdots dx_n.$$

Applying estimate (4) to the inner integral in x_1, we get

$$\left| \int e^{i\lambda\phi}\psi_j\,dx \right| \leq c_k(a_k\lambda)^{-1/k} \left(\|\psi\|_{L^\infty} + \int \left| \frac{\partial\psi}{\partial x_1}(x_1,\ldots,x_n) \right| dx_1 \right).$$

Finally, integrating over the remaining variables $x_2,\ldots,x_n$ yields the desired bound. $\square$

2.1.1 Proof of the Basis Claim

We now verify that the polynomials of the form

$$(\xi\cdot x)^k, \quad \xi\in\mathbb{R}^n,$$

span the space of real homogeneous polynomials of degree k. Define an inner product on this space by

$$\langle P, Q\rangle = \sum_{|\alpha|=k} \alpha!\,a_\alpha b_\alpha,$$

where $P(x)=\sum a_\alpha x^\alpha$, $Q(x)=\sum b_\alpha x^\alpha$. This inner product satisfies

$$\langle P, Q\rangle = [Q(\partial/\partial x)](P).$$

Suppose P is orthogonal to all $(\xi\cdot x)^k$. Then for all $\xi\in\mathbb{R}^n$,

$$(\xi\cdot\nabla)^k P = 0,$$

which implies

$$\left(\frac{d}{dt}\right)^k P(t\xi) = 0 \quad \text{for all } \xi.$$

This can only happen if $P\equiv 0$, proving that the set $(\xi\cdot x)^k$ spans the space.

Remark 8.

1. *In many instances, the estimate in* (15) *is not sharp. For example, for* $k=1$*, Proposition 6 gives a stronger bound. In* $\mathbb{R}^2$*, consider* $\phi(x)=x_1x_2$*:* (15) *only implies decay of order* $\lambda^{-1/2}$*, but the actual decay is* λ^{-1}*.*

2. *In one dimension, if* ϕ *has a critical point at* x_0 *and* ϕ' *does not vanish to infinite order there, then one can make a smooth change of variables to reduce* ϕ *near* x_0 *to a canonical form* $\tilde{\phi}(x)=\pm x^k$*. In higher dimensions, such a reduction is only generally possible in the special case* $k=2$*, which will be studied next.*

2.2 Nondegenerate Critical Points

Suppose ϕ has a critical point at x_0, and the Hessian matrix

$$H_{ij} = \left.\frac{\partial^2\phi}{\partial x_i \partial x_j}\right|_{x=x_0} \tag{16}$$

is invertible. Then x_0 is a nondegenerate critical point.

Proposition 9. *Suppose $\phi(x_0) = 0$, and ϕ has a nondegenerate critical point at x_0. If ψ is supported in a sufficiently small neighborhood of x_0, then*

$$\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)\,dx \sim \lambda^{-n/2}\sum_{j=0}^{\infty} a_j\lambda^{-j}, \quad \text{as } \lambda \to \infty, \tag{17}$$

where the asymptotic expansion is in the same sense as in earlier estimates.

Note. *Each constant a_j in the expansion depends only on finitely many derivatives of ϕ and ψ at x_0. Specifically,*

$$a_0 = \psi(x_0)\cdot(2\pi)^{n/2}\prod_{j=1}^{n}(-i\mu_j)^{-1/2},$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of the Hessian matrix of ϕ at x_0. The bounds in the error terms similarly depend only on upper bounds for finitely many derivatives of ϕ and ψ within the support of ψ.

Proof. The proof follows the same strategy as in Proposition 4.

First, consider the quadratic form

$$Q(x) = \sum_{j=1}^{m} x_j^2 - \sum_{j=m+1}^{n} x_j^2,$$

for some fixed $0 \le m \le n$. The analogue of the earlier estimate is

$$\int_{\mathbb{R}^n} e^{i\lambda Q(x)}e^{-|x|^2}x^\ell\,dx \sim \lambda^{-n/2-|\ell|/2}\sum_{j=0}^{\infty} c_j(m,\ell)\lambda^{-j}, \tag{18}$$

where $\ell = (\ell_1, \ldots, \ell_n)$ is a multi-index, $|\ell| = \sum \ell_j$, and $x^\ell = x_1^{\ell_1}\cdots x_n^{\ell_n}$. If any ℓ_j is odd, the integral vanishes.

This can be derived by expressing the integral as a product:

$$\prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{\pm i\lambda x_j^2}e^{-x_j^2}x_j^{\ell_j}\,dx_j = \prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{-x^2}x^{\ell_j}\,dx\cdot(1\mp i\lambda)^{-1/2-\ell_j/2},$$

then extracting the factor $\lambda^{-(n+|\ell|)/2}$ and expanding

$$\prod_{j=1}^{n}(\lambda^{-1} \mp i)^{-1/2-\ell_j/2}$$

in a power series in λ^{-1} for large λ.

An estimate analogous to the earlier bound is

$$\left|\int_{\mathbb{R}^n} e^{i\lambda Q(x)} x^\ell \eta(x)\, dx\right| \le A\lambda^{-n/2-|\ell|/2}, \tag{19}$$

for $\eta \in C_0^\infty(\mathbb{R}^n)$. To show this, consider the cones

$$\Gamma_j = \{x : |x_j|^2 \ge |x|^2/2n\}, \quad \Gamma_j^0 = \{x : |x_j|^2 \ge |x|^2/n\}.$$

Since $\bigcup_j \Gamma_j^0 = \mathbb{R}^n$, we construct smooth, degree-0 homogeneous functions Ω_j supported in Γ_j, with $\sum_j \Omega_j(x) = 1$ for all $x \neq 0$.

Then,

$$\int_{\mathbb{R}^n} e^{i\lambda Q(x)} x^\ell \eta(x)\, dx = \sum_j \int_{\mathbb{R}^n} e^{i\lambda Q(x)} x^\ell \eta(x)\Omega_j(x)\, dx.$$

In each cone Γ_j, apply integration by parts using

$$D_j e^{i\lambda Q(x)} = e^{i\lambda Q(x)}, \quad D_j f(x) = (\pm 2i\lambda x_j)^{-1} \cdot \frac{\partial f}{\partial x_j}.$$

Note that $|x_j| \ge (2n)^{-1/2}|x|$ in Γ_j, and

$$\left|({}^t D_j)^N \Omega_j(x)\right| \le C_N \lambda^{-N}|x|^{-2N},$$

which leads to the desired bound.

Also, if $\eta \in \mathcal{S}$ vanishes near the origin, then

$$\int e^{i\lambda Q(x)}\eta(x)\, dx = O(\lambda^{-N}), \quad \text{for all } N \ge 0. \tag{20}$$

Combining this with the above expansions yields the asymptotic formula in the special case $\phi(x) = Q(x)$.

To reduce the general case to this, apply Morse's lemma: Since $\phi(x_0) = \nabla\phi(x_0) = 0$ and the Hessian at x_0 is nondegenerate, there exists a diffeomorphism mapping a neighborhood of x_0 in x-space to one around 0 in y-space such that

$$\phi(y) = \sum_{j=1}^{m} y_j^2 - \sum_{j=m+1}^{n} y_j^2.$$

The index m corresponds to the number of positive eigenvalues of the Hessian matrix of ϕ at x_0.

To prove this, write

$$\phi(x) = \sum_{i,j} x_i x_j \phi_{ij}(x),$$

where $\phi_{ij} = \phi_{ji}$ are smooth and satisfy

$$\phi_{ij}(0) = \frac{1}{2} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(0).$$

This follows from the identity

$$\phi(x) = \int_0^1 \frac{d}{dt}[\phi(tx)]dt = \int_0^1 (1-t)\frac{d^2}{dt^2}[\phi(tx)]dt.$$

Assume inductively that

$$\phi(y) = \pm y_1^2 \pm \cdots \pm y_{r-1}^2 + \sum_{i,j \geq r} y_i y_j \tilde{\phi}_{ij}(y).$$

By a linear change of variables in $y_r, \ldots, y_n$, we may suppose $\phi_{rr}(0) \neq 0$, and hence $\phi_{rr}(y) \neq 0$ near 0.

Define new variables $y_1', \ldots, y_n'$ with $y_j' = y_j$ for $j \neq r$, and

$$y_r' = [\pm \phi_{rr}(y)]^{1/2} \left[y_r + \sum_{j>r} \frac{y_j \phi_{jr}(y)}{\pm \phi_{rr}(y)} \right].$$

Here the sign matches that of $\phi_{rr}(0)$, ensuring positivity.

This brings ϕ again to the same form with index increased by 1. Repeating the procedure, we reach $r = n$, proving the Morse lemma and hence reducing the general case to $\phi = Q$.

This completes the proof of the proposition. $\square$

3 Fourier Transforms of Measures Supported on Surfaces

Let S be an open subset of a smooth m-dimensional submanifold of $\mathbb{R}^n$. Denote by $d\sigma$ the surface measure on S induced by Lebesgue measure on $\mathbb{R}^n$, and let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a smooth cutoff function supported near S. We consider the finite Borel measure $d\mu = \varphi(x)\, d\sigma(x)$, supported on S, and we are interested in the decay of its Fourier transform:

$$\widehat{d\mu}(\xi) = \int_S e^{-2\pi i x\cdot\xi}\, \varphi(x)\, d\sigma(x). \tag{21}$$

This problem arises naturally in number theory, such as the distribution of lattice points in dilates of smooth domains.

3.1 The Sphere S^{n-1}

First we present a fundamental observation involving the decay of the Fourier transform of surface measure on the unit sphere. Let $S = \mathbb{S}^{n-1} \subset \mathbb{R}^n$. When $n > 1$, the Fourier transform of the surface measure $d\sigma$ on $\mathbb{S}^{n-1}$ is given by

$$\widehat{d\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-2\pi i x\cdot\xi}\, d\sigma(x), \tag{22}$$

and it exhibits a decay rate at infinity that is sharper than might initially be expected.

Theorem 10. *Let $\widehat{d\sigma}(\xi)$ be the Fourier transform of the surface measure on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then*

$$\widehat{d\sigma}(\xi) = 2\pi|\xi|^{(2-n)/2} J_{(n-2)/2}(2\pi|\xi|), \tag{23}$$

where $J_{(n-2)/2}$ is the Bessel function of the first kind of order $(n-2)/2$. Consequently,

$$|\widehat{d\sigma}(\xi)| = O\left(|\xi|^{(1-n)/2}\right), \quad \text{as } |\xi| \to \infty. \tag{24}$$

Proof. To prove the identity, without loss of generality, we may take

$$\xi = (0, 0, \ldots, 0, \xi_n), \quad \text{so that } |\xi| = \xi_n.$$

Then,

$$\widehat{d\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-2\pi i|\xi|\cos\theta}\, d\sigma(x),$$

where θ is the angle between the vector $x \in \mathbb{S}^{n-1}$ and the north pole direction $(0, \ldots, 0, 1)$.

We switch to spherical coordinates on $\mathbb{S}^{n-1}$. The surface measure in these coordinates takes the form

$$d\sigma = (\sin\theta)^{n-2}\, d\sigma_{n-2}\, d\theta,$$

where $d\sigma_{n-2}$ is the surface measure on the $(n-2)$-dimensional unit sphere, and so we write

$$\widehat{d\sigma}(\xi) = |\sigma_{n-2}| \int_0^\pi e^{-2\pi i|\xi|\cos\theta}(\sin\theta)^{n-2}\, d\theta,$$

where

$$|\sigma_{n-2}| = \int_{\mathbb{S}^{n-2}} d\sigma_{n-2}$$

is the total surface area of the $(n-2)$-sphere.

Setting the substitution $r = 2\pi|\xi|$ and $t = -\cos\theta$, and using the integral identity involving Bessel functions (as given in equation (12) of the text), we obtain the result

$$\widehat{d\sigma}(\xi) = 2\pi|\xi|^{(2-n)/2} J_{(n-2)/2}(2\pi|\xi|).$$

This concludes the proof. $\square$

3.2 Hypersurfaces with Nonzero Gaussian Curvature

We now consider a general hypersurface $S \subset \mathbb{R}^n$ of dimension $n-1$ which has nonzero Gaussian curvature at each point. More precisely, at any point $x_0 \in S$, we rotate and translate coordinates in $\mathbb{R}^n$ so that x_0 moves to the origin and the tangent hyperplane to S at x_0 becomes the hyperplane $x_n = 0$.

In a neighborhood of the origin (i.e., near x_0), the surface S may be expressed as the graph of a smooth function:

$$x_n = \phi(x_1, \ldots, x_{n-1}),$$

where $\phi \in C_0^\infty$, and satisfies

$$\phi(0) = 0, \quad \nabla\phi(0) = 0.$$

To define the Gaussian curvature, consider the Hessian matrix of second-order partial derivatives:

$$\left(\frac{\partial^2\phi}{\partial x_j \partial x_k}\right)(x_0).$$

The eigenvalues $\nu_1, \ldots, \nu_{n-1}$ of this matrix are known as the principal curvatures of the surface S at x_0. Their product, which equals the determinant of the above Hessian matrix, defines the Gaussian curvature (or total curvature) of S at x_0.

Theorem 11. *Suppose S is a smooth hypersurface in $\mathbb{R}^n$, whose Gaussian curvature is nonzero everywhere, and let $d\mu = \psi \, d\sigma$ be as above. Then*

$$|\widehat{d\mu}(\xi)| \leq A|\xi|^{(1-n)/2}. \tag{25}$$

Proof. For the proof, we first adopt a change in notation, replacing n with $n+1$ to align with the application of Proposition 9. By compactness, we may assume that S is represented locally as the graph

$$x_{n+1} = \phi(x_1, \ldots, x_n),$$

so that

$$d\sigma = \left(1 + |\nabla\phi|^2\right)^{1/2} \, dx_1 \cdots dx_n.$$

Thus, it suffices to show the estimate

$$\left| \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,\eta)} \widetilde{\psi}(x) \, dx \right| \leq A\lambda^{-n/2},$$

where $\lambda = |\xi| > 0$, $\xi = \lambda\eta$, and $\eta = (\eta_1, \ldots, \eta_{n+1}) \in \mathbb{S}^n$ is a unit vector. The phase function is given by

$$\Phi(x, \eta) = x \cdot \eta = \sum_{j=1}^{n} x_j \eta_j + \phi(x_1, \ldots, x_n)\eta_{n+1},$$

and we are given that

$$\phi(0) = 0, \quad \nabla\phi(0) = 0, \quad \det\left(\frac{\partial^2\phi}{\partial x_j \partial x_k}\right)(0) \neq 0.$$

We divide the proof into three cases based on the location of $\eta \in \mathbb{S}^n$:

1. η is near the north pole $\eta_N = (0, \ldots, 0, 1)$.

2. η is near the south pole $\eta_S = (0, \ldots, 0, -1)$.

3. η lies in the complement of these neighborhoods.

Case 1: η near η_N. We observe that $\nabla_x \Phi(x, \eta_N)|_{x=0} = 0$. For each η close to η_N, we want to find a unique point $x = x(\eta)$ so that

$$\nabla_x \Phi(x, \eta)|_{x=x(\eta)} = 0.$$

This condition defines n equations, and by the implicit function theorem, a solution exists provided

$$\det \left[\frac{\partial^2 \Phi}{\partial x_j \partial x_k} \right] (0, \eta_N) \neq 0.$$

This determinant equals the Gaussian curvature, which we assumed to be nonzero. For small enough neighborhoods around η_N, the determinant

$$\det \left[\frac{\partial^2 \Phi}{\partial x_j \partial x_k} \right] (x(\eta), \eta)$$

remains nonzero. Therefore, we may apply Proposition 9 with $x_0 = x(\eta)$, assuming $\widetilde{\psi}$ is supported in a sufficiently small neighborhood. Thus, the desired estimate holds in this region.

Case 2: η near η_S. This case is handled in the same way as Case 1, by symmetry.

Case 3: η in the complement. We compute

$$\nabla_x \Phi(x, \eta) = (\eta_1, \ldots, \eta_n) + \eta_{n+1} \nabla \phi(x).$$

Since η is bounded away from η_N and η_S, we have

$$\left(\eta_1^2 + \cdots + \eta_n^2\right)^{1/2} \geq c > 0.$$

Also,

$$\nabla \phi(x) = O(x) \quad \text{as } x \to 0.$$

Hence,

$$|\nabla_x \Phi(x, \eta)| \geq c' > 0$$

for x in a small neighborhood around the origin. Therefore, we may apply Proposition 6 to conclude that

$$\left| \int e^{i\lambda \Phi(x,\eta)} \widetilde{\psi}(x)\, dx \right| = O(\lambda^{-N}) \quad \text{for all } N \geq 0.$$

This completes the proof of the theorem.

□

3.3 Submanifolds of Finite Type

We now consider the decay of the Fourier transform in a more general setting. Let $S \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold, with $1 \leq m \leq n-1$. Instead of assuming nonvanishing Gaussian curvature, we impose the weaker requirement that S has only finite order of contact with any affine hyperplane at each point. Submanifolds satisfying this condition are said to be *of finite type*.

To make this precise, we proceed as follows. Let S be locally given near a point by a smooth embedding

$$\phi : U \subset \mathbb{R}^m \to \mathbb{R}^n,$$

where U is a neighborhood of the origin. We require that the differential

$$\left\{ \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_m} \right\}$$

is linearly independent at each point $x \in U$, so that ϕ defines a smooth embedding of U into $\mathbb{R}^n$.

Fix a point $x_0 \in U$, and let $\eta \in \mathbb{R}^n$ be a unit vector. We consider the scalar function

$$[\phi(x) - \phi(x_0)] \cdot \eta.$$

We say that S is of finite type at x_0 if, for each unit vector $\eta \in \mathbb{R}^n$, this function does not vanish to infinite order as $x \to x_0$; that is, there exists a multi-index α with $|\alpha| \geq 1$ such that

$$\partial_x^\alpha [\phi(x) \cdot \eta]|_{x=x_0} \neq 0.$$

Furthermore, the property is stable under small perturbations: if (x', η') is sufficiently close to (x_0, η), then

$$\partial_x^\alpha [\phi(x) \cdot \eta']|_{x=x'} \neq 0$$

still holds.

We define the *type* of the submanifold S at the point x_0 as the smallest integer $k \geq 1$ such that, for every unit vector $\eta \in \mathbb{R}^n$, there exists a multi-index α with $|\alpha| \leq k$ and

$$\partial_x^\alpha [\phi(x) \cdot \eta]|_{x=x_0} \neq 0.$$

If $U_1 \subset U$ is compact, then the type of ϕ in U_1 is the maximum of the types at points $x_0 \in U_1$.

Remark 12 (Examples of Finite Type Submanifolds)**.** *The following may help to understand what are finite type submanifolds:*

1. *If S is a curve in $\mathbb{R}^2$, the finite type condition is equivalent to the curvature not vanishing to infinite order at x_0.*
2. *If S is a curve in $\mathbb{R}^3$, the condition is equivalent to neither the curvature nor the torsion vanishing to infinite order at x_0.*
3. *If S is a hypersurface in $\mathbb{R}^n$, the condition holds if at least one of the principal curvatures does not vanish to infinite order at x_0.*
4. *If S is real-analytic, then the condition is equivalent to the submanifold not being entirely contained in any affine hyperplane.*

Theorem 13. *Suppose S is a smooth m-dimensional manifold in $\mathbb{R}^n$ of finite type. Let $d\mu = \psi\, d\sigma$ be as above. Then*

$$|\widehat{d\mu}(\xi)| \leq A|\xi|^{-1/k}, \tag{26}$$

where k is the type of S inside the support of ψ.

Proof. By applying a suitable partition of unity, we reduce to the local case, where the integral becomes

$$\int_S e^{-2\pi i x\cdot\xi}\, d\mu(x) = \int_{\mathbb{R}^m} e^{-2\pi i\phi(x)\cdot\xi}\, \widetilde{\psi}(x)\, dx,$$

for some $\widetilde{\psi} \in C_0^\infty$, with support as small as we like.

We can write $\xi = \lambda\eta$, where $\lambda > 0$ and $|\eta| = 1$. By the finite type condition, there exists a multi-index α with $|\alpha| \leq k$ such that

$$\partial_x^\alpha[\phi(x)\cdot\eta] \neq 0$$

for all $x \in \operatorname{supp}(\widetilde{\psi})$, provided this support is sufficiently small.

Thus, we can apply Proposition 7 (specifically estimate (15)) to conclude that

$$\left|\int_{\mathbb{R}^m} e^{-2\pi i\phi(x)\cdot\xi}\, \widetilde{\psi}(x)\, dx\right| \leq C\lambda^{-1/k} = C|\xi|^{-1/k},$$

which establishes the theorem. $\square$

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